ON THE REFLECTION OF MAGNETO-ACOUSTIC WAVES

(OB OTRIAZHENII MAGNITOZVUKOVYKH VOLN)

PMM Vol.26, No.5, 1962, pp. 842-847

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(Received March 17, 1962)

This paper deals with the problem of reflection of magneto-acoustic waves from a plane boundary between a fluid and an elastic medium conducting electricity. Expressions are obtained for the amplitude coefficients of reflection and refraction. The surface waves on a free boundary of an elastic medium are considered. The velocity of these waves is determined for the case of a weak magnetic field.

1. Magnetohydrodynamic and magneto-elastic waves. Let a fluid and an elastic conducting medium be placed in a uniform constant magnetic field $\mathbf{H}$. In order to solve the problem of reflection of waves from the boundary between these two media, it is necessary to consider the propagation of waves separately in each medium.

The linearized equations of magnetohydrodynamics for plane waves with the vector $\mathbf{k}$ and frequency ω reduce to the system of algebraic equations [1]

$$
\begin{gathered}
-\omega \mathbf{h} = \mathbf{k}\times(\mathbf{v}\times\mathbf{H}) + i u_0^2 \eta_0 k^2 \mathbf{h} \\
-\omega \mathbf{v} + \frac{u_0^2}{\omega}\mathbf{k}\,(\mathbf{k}\mathbf{v}) = -\frac{1}{4\pi\rho_0}\mathbf{H}\times(\mathbf{k}\times\mathbf{h}) \qquad (1.1) \\
\eta_0 = \frac{c^2}{4\pi\sigma_0 u_0^2}
\end{gathered}
$$

where $\mathbf{v}$ is the velocity of the fluid, $\mathbf{h}$ is the small variation of the magnetic field in the wave, ρ_0 and σ_0 are the density and electrical conductivity of the fluid, u_0 denotes the velocity of sound in the fluid, and c is the velocity of light.

The plane boundary between the two media will be designated as the plane xy. Assuming that the vectors $\mathbf{H}$ and $\mathbf{k}$ are in the plane xz, we write Equations (1.1) in terms of components

$$
\begin{aligned}
-\omega h_x &= k_z (v_x H_z - v_z H_x) + i u_0^2 \eta_0 k^2 h_x \\
-\omega h_z &= k_x (v_z H_x - v_x H_z) + i u_0^2 \eta_0 k^2 h_z \\
-\omega v_x + \frac{u_0^2}{\omega} k_x (\mathbf{kv}) &= -\frac{H_z}{4\pi\rho_0} (k_x h_z - k_z h_x) \\
-\omega v_z + \frac{u_0^2}{\omega} k_z (\mathbf{kv}) &= -\frac{H_x}{4\pi\rho_0} (k_z h_x - k_x h_z)
\end{aligned}
\tag{1.2}
$$

The condition of compatibility of this system has the form

$$
\begin{gathered}
u^2 - (1 + \psi_0) u + \psi_0 (\mathbf{k}^\circ \mathbf{H}^\circ)^2 + i\omega\eta_0 (u - 1) = 0 \\
u = \left(\frac{\omega}{k u_0}\right)^2, \qquad \psi_0 = \frac{H^2}{4\pi\rho_0 u_0^2}
\end{gathered}
\tag{1.3}
$$

Here, u and ψ_0 are the squares of the phase velocity and the intensity of the magnetic field in the dimensionless form; $\mathbf{k}^\circ$ and $\mathbf{H}^\circ$ are the magnitudes of the vectors $\mathbf{k}$ and $\mathbf{H}$. Equation (1.3) has, for small $\omega\eta_0$, the roots u_1 and u_2 corresponding to the fast and the slow magneto-acoustic waves. According to (1.2), we have for these waves

$$
\begin{gathered}
v_{\nu x} = M_\nu v_{\nu z}, \qquad M_\nu = -\frac{k_{\nu x} k_{\nu z} k_\nu^{-2} - \psi_\nu H_x H_z H^{-2}}{-u_\nu + k_{\nu x}^2 k_\nu^{-2} + \psi_\nu H_z^2 H^{-2}} \\
\psi_\nu = \psi_0 \left(1 + i\,\frac{\omega\eta_0}{u_0}\right)^{-1} \qquad (\nu = 1, 2)
\end{gathered}
\tag{1.4}
$$

The equations of plane waves in an elastic medium are [2]

$$
\begin{gathered}
-\omega \mathbf{h} = \mathbf{k}\times(\mathbf{v}\times\mathbf{H}) + i a^2 \eta k^2 \mathbf{h} \\
-\frac{\omega^2}{a^2}\mathbf{v} = -\mathbf{k}(\mathbf{kv}) + \xi \mathbf{k}\times(\mathbf{k}\times\mathbf{v}) - \frac{\omega\psi}{H^2} \mathbf{H}\times(\mathbf{k}\times\mathbf{h}) \\
a^2 = \frac{\lambda + 2\mu}{\rho}, \quad b^2 = \frac{\mu}{\rho}, \quad \xi = \frac{b^2}{a^2}, \quad \eta = \frac{c^2}{4\pi\sigma a^2}; \quad \psi = \frac{H^2}{4\pi\rho a^2}
\end{gathered}
\tag{1.5}
$$

Here, a^2 and b^2 are the squares of the velocities of transverse and longitudinal waves, respectively; ρ, λ and μ are the density and the Lamé constants of the elastic medium; σ is the electrical conductivity of the elastic medium.

We write Equations (1.5) in terms of components

$$
\begin{aligned}
-\omega h_x &= k_z (v_x H_z - v_z H_x) + i a^2 \eta k^2 h_x \\
-\omega h_z &= k_x (v_z H_x - v_x H_z) + i a^2 \eta k^2 h_z
\end{aligned}
$$

$$\left(-\frac{\omega^2}{a^2}+k_x^2+\xi k_z^2\right)v_x+(1-\xi)\,k_xk_zv_z=-\frac{\omega\psi H_z}{H^2}(k_xh_z-k_zh_x) \tag{1.6}$$
$$\left(-\frac{\omega^2}{a^2}+k_z^2+\xi k_x^2\right)v_z+(1-\xi)\,k_xk_zv_x=-\frac{\omega\psi H_x}{H^2}(k_zh_x-k_xh_z)$$

The condition of compatibility of this system has the form

$$u^2-(1+\xi+\psi)\,u+\xi+\psi\,[(\mathbf{k}^\circ\mathbf{H}^\circ)^2+\xi\,(\mathbf{k}^\circ\times\mathbf{H}^\circ)^2]+i\,\frac{\omega\eta}{u}\,(u-1)\,(u-\xi)=0$$
$$u=\left(\frac{\omega}{ka}\right)^2 \tag{1.7}$$

For small $\omega\eta$, two roots of this equation, u_3 and u_4, correspond to the fast and the slow magneto-elastic waves, while the third root corresponds to an aperiodic process.

From Equations (1.6) we obtain the relations between the components of the velocity of the medium in magneto-elastic waves

$$v_{\nu x}=M_\nu v_{\nu z},\qquad M_\nu=-\frac{(1-\xi)\,k_{\nu x}k_{\nu z}\;{}_{\nu}^{-2}-\psi_\nu H_xH_zH^{-2}}{-u_\nu+(k_{\nu x}^2+\xi k_{\nu z}^2)\,k_\nu^{-2}+\psi_\nu H_z^2H^{-2}}$$
$$\psi_\nu=\psi\left(1+i\,\frac{\omega\eta}{u_\nu}\right)^{-1}\qquad(\nu=3,\,4) \tag{1.8}$$

2. Reflection of magneto-acoustic waves. We shall assume that both media are perfect conductors ($\eta_0=\eta=0$). On the boundary between the two media, the normal component of stress, the normal component of the velocity, the magnetic field, and the tangential component of the electric field should be continuous. The last condition implies, for perfect conductivity, the continuity of the tangential component of the velocity of the medium.

Therefore, the boundary conditions are

$$[v_z]=0,\qquad [v_x]=0,\qquad P_{zz}=-p,\qquad P_{xz}=0 \tag{2.1}$$

where $[v_i]$ is the jump of the quantity v_i on the boundary, P_{zz} and P_{xz} are the components of the stress tensor in the elastic medium, and p is the pressure in the fluid. In the case of monochromatic waves, they can be expressed in terms of the components of the velocities of the elastic and the fluid medium in the following way

$$P_{zz}=\frac{i}{\omega}\,\rho a^2\left(\operatorname{div}v+2\xi\,\frac{\partial v_z}{\partial z}\right),\qquad P_{xz}=2\,\frac{i}{\omega}\,\rho b^2\left(\frac{\partial v_x}{\partial z}+\frac{\partial v_z}{\partial x}\right)$$
$$p=-\frac{i}{\omega}\rho_0u_0^2\operatorname{div}\mathbf{v}$$

Let a fast magneto-acoustic wave fall on the boundary (Fig. 1). The quantities describing the incident waves will be denoted by primes. For the velocity field in the fluid, according to (1.4), we have

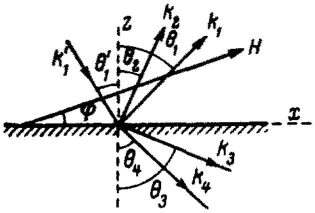

Fig. 1.

$$v_z = v_{1z}' + v_{1z} + v_{2z}$$
$$v_x = M_1' v_{1z} + M_1 v_{1z} + M_2 v_{2z} \tag{2.2}$$
$$M_1' = \frac{1}{2}\frac{\sin 2\theta_1' + \psi_0 \sin 2\varphi}{-u_1' + \sin^2\theta_1' + \psi_0 \sin^2\varphi}$$
$$M_\nu = -\frac{1}{2}\frac{\sin 2\theta_\nu - \psi_0 \sin 2\varphi}{-u_\nu + \sin^2\theta_\nu + \psi_0 \sin^2\varphi} \qquad (\nu = 1, 2)$$

For the velocity field in the elastic medium, according to (1.8)

$$v_z = v_{3z} + v_{4z}, \qquad v_x = M_3 v_{3z} + M_4 v_{4z}$$
$$M_\nu = \frac{1}{2}\frac{(1-\xi)\sin 2\theta_\nu + \psi \sin 2\varphi}{-u_\nu + \sin^2\theta_\nu + \xi\cos^2\theta_\nu + \psi\sin^2\varphi} \qquad (\nu = 3, 4) \tag{2.3}$$

Taking into account that

$$v_{1z}' = A_1' \cos\beta_1' \exp i\,[(\mathbf{k}_1'\mathbf{r}) - \omega t], \qquad \tan\beta_1' = M_1'$$
$$v_{\nu z} = A_1' W_\nu \cos\beta_\nu \exp i\,[(\mathbf{k}_\nu\mathbf{r}) - \omega t], \qquad \tan\beta_\nu = M_\nu \tag{2.4}$$
$$(\nu = 1, 2, 3, 4)$$

where β_ν is the angle between the displacement vector in the wave with index ν and the axis z, from the boundary conditions (2.1) we obtain a system of four equations for the amplitude coefficients of reflection and refraction W_ν.

The solutions of this system have the form

$$W_1 = -\frac{\cos\beta_1'}{\cos\beta_1}\frac{\rho a^2(M_2 - M_1')X + \rho_0 u_0^2 B}{\rho a^2(M_2 - M_1)X + \rho_0 u_0^2 D} \tag{2.5}$$
$$W_3 = \frac{M_4\cot\theta_4 - 1}{(M_2 Z - Y)\cos\beta_3}\left[(M_2 - M_1')\cos\beta_1' + (M_2 - M_1)W_1\cos\beta_1\right]$$
$$W_4 = \frac{\cos\beta_3}{\cos\beta_4}\frac{1 - M_3\cot\theta_3}{M_4\cot\theta_4 - 1}W_3$$
$$W_2 = \frac{\cos\beta_1' - W_1\cos\beta_1 - W_3\cos\beta_3 - W_4\cos\beta_4}{\cos\beta_2}$$

$$X = (M_4\cot\theta_4 - 1)[\cot\theta_3 - (1-2\xi)M_3] + (1 - M_3\cot\theta_3)[\cot\theta_4 - (1-2\xi)M_4]$$
$$Y = M_4 - M_3 + M_3 M_4(\cot\theta_4 - \cot\theta_3)$$
$$Z = M_4\cot\theta_4 - M_3\cot\theta_3$$
$$B = Y(M_2 - M_1' + \cot\theta_2 + \cot\theta_1') - Z(M_2\cot\theta_1' + M_1'\cot\theta_2)$$
$$D = Y(M_2 - M_1' + \cot\theta_2 - \cot\theta_1') + Z(M_2\cot\theta_1' - M_1'\cot\theta_2)$$

Assuming $\psi_0 = \psi = 0$ and considering that for this case

$$u_1' = u_1 = 1, \quad u_2 = 0, \quad \theta_2 = 0, \quad u_3 = 1, \quad u_4 = \xi$$

$$M_1' = -M_1 = -\tan\theta_1, \quad M_2 = -\cot\theta_2, \quad M_3 = -\tan\theta_3, \quad M_4 = \cot\theta_4$$

$$\beta_1' = \pi - \theta_1, \quad \beta_1 = \theta_1, \quad \beta_3 = \pi - \theta_3, \quad \beta_4 = \frac{\pi}{2} - \theta_4$$

we obtain from the Formula (2.5) the expressions for the coefficients of reflection and refraction [3, p.31] without magnetic field.

Increasing ψ_0 and ψ to infinity, we have

$$u_1' = u_1 = \psi_0, \quad u_3 = \psi, \quad \theta_2 = \theta_4 = 0$$

$$M_1' = M_1 = M_3 = -\tan\varphi$$

$$M_2 = M_4 = \cot\varphi$$

$$\beta_1' = \beta_1 = \beta_3 = \pi - \varphi$$

$$W_1 = -1, \quad W_2 = W_3 = W_4 = 0$$

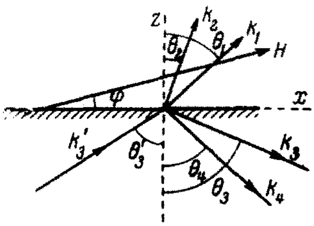

Fig. 2.

i.e. the reflection is complete in this limit case.

Let now a fast magneto-elastic wave fall from the elastic medium on the boundary surface (Fig. 2).

The velocity field in the fluid is

$$v_z = v_{1z} + v_{2z}, \qquad v_x = M_1 v_{1z} + M_2 v_{2z}$$

and the velocity field in the elastic medium is

$$v_z = v_{3z}' + v_{3z} + v_{4z}, \qquad v_x = M_3' v_{3z}' + M_3 v_{3z} + M_4 v_{4z}$$

$$M_3' = -\frac{1}{2}\,\frac{(1-\xi)\sin 2\theta_3' - \psi\sin 2\varphi}{-u_3' + \sin^2\theta_3' + \xi\cos^2\theta_3' + \psi\sin^2\varphi}$$

As in the preceding problem, we find

$$W_3 = -\frac{\cos\beta_3'}{\cos\beta_3}\,\frac{\rho a^2 (M_2 - M_1) F + \rho_0 u_0^2 E}{\rho a^2 (M_2 - M_1) X + \rho_0 u_0^2 \Phi} \tag{2.6}$$

$$W_4 = \frac{(1 + M_3'\cot\theta_3')\cos\beta_3' + (1 - M_3\cot\theta_3) W_3\cos\beta_3}{(M_4\cot\theta_4 - 1)\cos\beta_4}$$

$$W_1 = \frac{(M_2 - M_3')\cos\beta_3' + (M_2 - M_3) W_3\cos\beta_3 + (M_2 - M_4) W_4\cos\beta_4}{(M_2 - M_1)\cos\beta_1}$$

$$W_2 = \frac{\cos\beta_3' - W_1\cos\beta_1 + W_3\cos\beta_3 + W_4\cos\beta_4}{\cos\beta_2}$$

$$F = (1 + M_3' \cot \theta_3')\,[\cot \theta_4 - (1 - 2\xi) M_4] -$$
$$- (M_4 \cot \theta_4 - 1)\,[\cot \theta_3' + (1 - 2\xi) M_3']$$
$$L = M_4 - M_3' + M_3' M_4 (\cot \theta_4 + \cot \theta_3')$$
$$N = M_4 \cot \theta_4 + M_3' \cot \theta_3'$$
$$E = L (M_2 - M_1 + \cot \theta_2 - \cot \theta_1) + N (M_2 \cot \theta_1 - M_1 \cot \theta_2)$$
$$\Phi = Y (M_2 - M_1 + \cot \theta_2 - \cot \theta_1) + Z (M_2 \cot \theta_1 - M_1 \cot \theta_2)$$

For $\psi_0 = \psi = 0$, we obtain from (2.6) the known expressions for the coefficients W_ν [3, p.34].

If ψ_0 and ψ increase to infinity we have, as it was previously, the case of complete reflection

$$M_3' = M_3 = M_1 = -\tan \varphi, \qquad \beta_3' = \beta_3 = \beta_1 = \pi - \varphi$$
$$W_3 = -1, \qquad W_1 = W_2 = W_4 = 0$$

3. Surface waves (Rayleigh). We shall investigate the effect of a magnetic field on the surface waves, assuming that the elastic medium is adjacent to a sufficiently rarefied gaseous medium ($\rho_0 = 0$). It is known that the equations of the surface waves can be obtained by increasing to infinity the coefficient of reflection of plane waves [3]. In this way we obtain from (2.6)

$$X = 0 \tag{3.1}$$

Let us consider a weak magnetic field ($\psi \ll 1$). With the accuracy up to the terms linear in ψ we have

$$u_3 = 1 + \psi \cos^2 (\theta_3 - \varphi), \qquad u_4 = \xi + \psi \sin^2 (\theta_4 - \varphi)$$
$$M_3 = -\tan \theta_3 \left[1 + \frac{\psi}{1 - \xi} \left(\frac{\sin 2\varphi}{\sin 2\theta_3} - \frac{\cos^2 (\theta_3 - \varphi) - \sin^2 \varphi}{\cos^2 \theta_3}\right)\right] \tag{3.2}$$
$$M_4 = \cot \theta_4 \left[1 + \frac{\psi}{1 - \xi} \left(\frac{\sin 2\varphi}{\sin 2\theta_4} + \frac{\sin^2 (\theta_4 - \varphi) - \sin^2 \varphi}{\sin^2 \theta_4}\right)\right]$$

Introducing the notations

$$q = \frac{u_4}{u_3}, \qquad S = \sin^2 \theta_4 \tag{3.3}$$

and taking into account

$$\sin \theta_3 = (\sin \theta_4) \,/\, \sqrt{q}$$

we reduce (3.1) to the form

$$1+\frac{(1-2S)^2}{4S\sqrt{1-S}\sqrt{\xi-S}}=\frac{\psi}{4\xi(1-\xi)(1-2S_0)}\left\{\left[2\alpha+\frac{\xi^2-(1-2\xi)(1-2S_0)}{\xi-S_0}\beta\right]\cos 2\varphi-\right.$$
$$\left.-\frac{1}{\sqrt{S_0(1-S_0)}}\left[\alpha(1-2S_0)+2\beta S_0(1-S_0)\left(\frac{1-\xi}{\xi-S_0}-\frac{2\xi}{1-2S_0}\right)\right]\sin 2\varphi+\frac{\beta(1-\xi)^2}{\xi-S_0}\right\} \qquad (3.4)$$

Here, $S_0 > 1$ is the real root of this equation for $\psi = 0$, and

$$\alpha = 1-2S_0(1-\xi),\quad \beta=\xi-2S_0(1-\xi)$$

The solution of Equation (3.4) is

$$S=S_0\left[1+\frac{(1-S_0)(\xi-S_0)A\psi}{2\xi(1-\xi)[2(\xi-S_0)-S_0(\alpha-\xi)]}\right] \qquad (3.5)$$

where A denotes the term in the braces in the right-hand side of equation (3.4).

The phase velocity of surface waves along the boundary is determined by the formula

$$v=a\sqrt{\frac{u_1}{S}}=v_0\left\{1+\frac{\psi}{4\xi}\left[1-(1-2S_0)\cos 2\varphi-2\sqrt{S_0(1-S_0)}\sin 2\varphi-\right.\right.$$
$$\left.\left.-\frac{(1-S_0)(\xi-S_0)A}{(1-\xi)[2(\xi-S_0)-S_0(\alpha-\xi)]}\right]\right\} \qquad \left(v_0=\frac{b}{\sqrt{S_0}}\right)$$

As it follows from (3.6), v assumes real values only for $\varphi = 0$ and $\varphi = \pi/2$. This indicates that a surface wave propagates without attenuation in the cases of the magnetic field being parallel or perpendicular to the surface. For other directions of the magnetic field v is a complex quantity and the surface waves are damped. This damping can be explained by the fact that for $0 < \varphi < \pi/2$ an electromagnetic wave is generated which continuously absorbs part of the energy of the surface wave. The coefficient of attenuation is equal to $\kappa = \mathrm{Im}(\omega/v)$.

We shall write the expressions for v in the following cases:

when $\varphi = 0$

$$v=v_0\left\{1+\frac{\psi S_0}{2\xi}\left[1-\frac{1-S_0}{1-\xi}\,\frac{(1-\xi)(\xi-S_0)+(1+\xi-2S_0)\xi\beta}{S_0[2(\xi-S_0)-S_0(\alpha-\xi)]}\right]\right\}$$

when $\varphi = \pi/2$

$$v=v_0\left\{1+\frac{\psi(1-S_0)}{2\xi}\left[1+\frac{(1-\xi)(\xi-S_0)-(1-3\xi+2S_0\xi)\beta}{(1-\xi)[2(\xi-S_0)-S_0(\alpha-\xi)]}\right]\right\}$$

when $\varphi = \pi/4$

$$v = v_0\left\{1 + \frac{\psi[\xi - S_0 + \xi(1-S_0)\beta]}{4\xi[2(\xi - S_0) - S_0(\alpha - \xi)]} \times\right.$$

$$\left.\times\left[1 + \frac{i(\xi - S_0)\sqrt{S_0(S_0-1)}\,[(1-2S_0)^2 - 4S_0(1-S_0)(\alpha - \xi + \xi\beta)]}{S_0(1-\xi)(1-2S_0)[\xi - S_0 + \xi(1-S_0)\beta]}\right]\right\}$$

The values of v/v_0 for two values of ξ are as follows

$\varphi = 0$	$\pi/2$	$\pi/4$	
$v/v_0 = 1 + 1.87\psi$	1	$1 + (0.93 - i0.32)\psi$	for $\xi = 1/3$
$v/v_0 = 1 + 1.26\psi$	1	$1 + (0.61 - i0.28)\psi$	for $\xi = 1/2$

Thus, a magnetic field parallel to the surface increases slightly the velocity of surface waves, while a magnetic field perpendicular to the surface practically does not influence this velocity.

The coefficient of attenuation κ, for $\varphi = \pi/4$, is equal to

$$\varkappa = 0.32\omega\psi / v_0 \qquad \text{for } \xi = 1/3$$
$$\varkappa = 0.28\omega\psi / v_0 \qquad \text{for } \xi = 1/2$$

The author thanks K.P. Staniukovich who proposed the problem of this paper.

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Translated by M.P.B.